北星女子短大紀要 *Journal of Hokusei Jr. Col. Vol. 35 : 193－204, 1999*

On Sufficient Conditions of a Measure-Theoretic Probability Model of Measurements Describing Quantum-Mechanical Probability

測定の測度論的確率模型が量子学的確率を記述できるための十分条件について

Satoshi Uchiyama
内 山 　 智

要 約

有限次元量子力学的確率空間を導くような測度論的確率論に基づく測定の一般的な模型が提案された。この模型の根元事象は対象とその環境の結合系の状態で表される。事象は、対応した測定過程の曲線より成る。この確率空間から量子力学的確率空間を導出するには、3つの自明ではない前提で十分であることが示された。第1前提は、どんな測定も対象の状態を他のものに変化させるということを意味している。理想的測定による対象の状態の無限小変化は、対象の状態空間$T_{\xi_0}\Gamma$への接ベクトルで表される。量子力学的状態ベクトルと同一視される。このように$\xi_0 \in \Gamma$に於ける接ベクトル空間$T_{\xi_0}\Gamma$は、量子力学的確率空間のヒルベルト空間と同一視される。また、測定器の正当性が、それによる理想的測定による対象の状態の無限小変化達の直交性を含意するということも示された。

キーワード：量子力学、測度論的確率論、量子力学的確率

ABSTRACT

A general model of measurements based on the measure-theoretic probability theory that leads to a finite-dimensional quantum-mechanical probability space is proposed. Elementary events of the model are represented by states of the compound system consisting of the object and the environment. Events consist of curves of the corresponding measuring processes. It is shown that three nontrivial premises are sufficient to derive a quantum-mechanical probability space from the probability space. The first premise means that any measurement changes the state of the object into another one. An infinitesimal variation of the state of the object under an ideal measurement is represented by a tangent vector to the state space Γ of the object. It is identified with a quantum-mechanical state vector. Thus the tangent space $T_{\xi_0}\Gamma$ at a certain $\xi_0 \in \Gamma$ is identified with the Hilbert space of a quantum-mechanical probability space. It is also shown that the validity of the measuring apparatus implies the orthogonality of the infinite variations of states of the object under the ideal measurement.

1. Introduction

Probabilistic behavior of events is usually considered to be due to our lack of knowledge. Such a concept of probability is formulated mathematically as measure-theoretic probability theory. Probability in quantum phenomena, however, is not described by a single measure-theoretic probability space, if we make quantum-mechanical observables correspond to random variables on the measure-theoretic probability space. This is known as the no-go theorem.[1] This is the reason why some people consider that probability is not due to our lack of knowledge but is one of the essentials of Nature.

But if there is a possibility of probability in quantum-phenomena being described within the framework of measure-theoretic probability theory, then it is possible for us to understand probabilistic behavior of quantum phenomena at a deeper level. In this paper, we will investigate how measure-theoretic probability theory leads to a finite-dimensional quantum-mechanical probability space.

Section 2 is devoted to formulating finite-dimensional quantum-mechanical probability spaces. In section 3, we will clarify the sufficient conditions under which a general model of measurements for a physical object based on the measure-theoretic probability theory leads to a quantum-mechanical probability space. It will be shown that three nontrivial premises are used to derive the quantum-mechanical probability space.

2. Quantum-mechanical probability space

There are several mathematical formulations of quantum-mechanical probability.[2,3,4] A formulation of quantum-mechanical probability theory that is sufficient for our purpose is provided in this section. The formulation has formal analogies with measure-theoretic probability theory,[5] but it contains new notions which are not contained in the latter. The aim of this section is not to develop a general theory of quantum probability, but to fix terminology in order to avoid the expected confusion. Since the formulation is concerned with only general aspects of quantum mechanics, I think that it is consistent with other formulations.

A *quantum measurable space* (hereafter,QMS) $(\mathcal{H},\mathcal{P})$ consists of two objects: A Hilbert space $\mathcal{H}$ and a nonempty set $\mathcal{P}$ of projections on $\mathcal{H}$. In the case that $\mathcal{H}$ is a complex (real) n-dimensional Hilbert space, we call $(\mathcal{H},\mathcal{P})$ a complex (real) n-dimensional quantum measurable space.

$\mathcal{H}$ is an analogue of a set of elementary events, and $\mathcal{P}$ is an analogue of a set of events in measure-theoretic probability theory. $\mathcal{P}$ is not always the set of all projections on $\mathcal{H}$, if a superselection rule [2,6] exists. Even if no superselection rule exists, we should restrict $\mathcal{P}$ to a set of projections that can have explicit physical meaning, because we do not know generally how to measure a quantity represented by an arbitrary self-adjoint operator.[7] $\mathcal{P}$ is usually assumed to form an orthomodular lattice.[2]

The properties of a QMS depend on its dimension. Two-dimensional QMSs, infinite-dimensional QMSs and n-dimensional QMSs ($3 \leq n < \infty$) have different features.[8] In particular, treatment in the infinite-dimensional case, that is the case of usual quantum mechanics, is technically rather complicated; [9] what we are to argue, however does not relate to the specifics of the infinite dimension. Thus we restrict our consideration to finite-dimensional cases in this paper.

We will say that two complete orthonormal systems (hereafter, CONSs) $\{u_i\}$ and $\{v_i\}$ of $\mathcal{H}$ are *equivalant*, if and only if there exists a permutation σ of $\{1,2,\ldots,\dim \mathcal{H}\}$ and scalars ζ_i's such that $u_i = \zeta_i v_{\sigma(i)}$ We fix a representative of each equivalence class of CONSs, and we will refer an equivalent class by its fixed representative.

In an n-dimensional QMS $(\mathcal{H},\mathcal{P})$, taking an orthogonal set of n projections in P whose ranges are one-dimensional subspaces, we can construct a CONS by picking out a unit vector from the ranges of each projection. We call such a CONS (more precisely, the equivalence class) a *context*. We denote the set of all contexts by $\mathcal{C}$.

When a linear operator on $\mathcal{H}$ is diagonalized in a CONS that is a representative of a context $\underline{\alpha} \in \mathcal{C}$, we say that the linear operator *belongs* to the context $\underline{\alpha}$. A linear combination with real coefficients of projections in $\mathcal{P}$ that belong to a context is called an *observable*. The set of all observables is denoted by $\mathcal{O}$.

As pointed out by Wiener and Siegel,[10] degenerate operators play a subtle role in hidden-variable interpretation. Hence we introduce a similar notion: In a QMS $(\mathcal{H},\mathcal{P})$, an observable $C \in \mathcal{O}$ is said to be *degenerate within* $\mathcal{C}$, if and only if C belongs to different contexts $\underline{\alpha}$, $\underline{\beta} \in \mathcal{C}$ simultaneously. The zero and the identity are *trivially degenerate observables*.

A simple example of a degenerate observable within $\mathcal{C}$ is given in Ref. 11, Chap. 6, Sec. 5, p.121. Another simple example, though artificial, is the following: $\mathcal{C}$ consists of $\underline{\alpha}$ and $\underline{\beta}$, where $\underline{\alpha} = \{(1,0,0)^T,\ (0,1,0)^T,\ (0,0,1)^T\}$ and $\underline{\beta} = \{(1,0,0)^T,\ (0,1/\sqrt{2},1/\sqrt{2})^T,\ (0,1/\sqrt{2},\ -1/\sqrt{2})^T\}$; Put

$$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix},\ B=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\ C=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix};\qquad (1)$$

A and B belong to $\underline{\alpha}$ and $\underline{\beta}$, respectively, but C belongs to the both contexts, i.e., C is degenerate within $\mathcal{C}$.

A *quantum-mechanical probability space* (hereafter, QMPS) $(\mathcal{H},\mathcal{P},\ \Psi)$ consists of a QMS $(\mathcal{H},\mathcal{P})$ and a density operator Ψ, which satisfies $\Psi^{\dagger} = \Psi$, $\Psi \geq 0$ and $\mathrm{Tr}\Psi = 1$. Ψ is called a *quantum-mechanical state*. Ψ is an analogue of a probability measure in measure-theoretic probability theory.

The minimal interpretation of this mathematical formulation of a QMPS $(\mathcal{H}, \mathcal{P}, \Psi)$ is given as follows:

1. An outcome of a measurement of a physical quantity that is represented by an observable $A \in \mathcal{O}$ belongs to the spectrum of A.
2. *The expectation value of the outcome of a measurement of* $A \in \mathcal{O}$ *performed on a* quantum-mechanical state Ψ is given by

$$\mathrm{E}_{\Psi}(A) := \mathrm{Tr}(A\Psi). \tag{2}$$

If a quantum-mechanical state is pure, then by taking the corresponding unit vector Ψ in $\mathcal{H}$ instead of a density operator, (2) can be written as

$$\mathrm{E}_{\Psi}(A) = \langle \Psi, A\Psi \rangle, \tag{3}$$

where $\Psi \in \mathcal{H}$, $\|\Psi\|^2 := \langle \Psi, \Psi \rangle = 1$, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$ which is antilinear in the first entry, linear in the second entry.

3. A general model of measurements

In this section, a general model of measurements based on a measure-theoretic probability space is formulated and it is shown how the model has quantum-mechanical probability space structure.

Let Γ be an m-dimensional analytic manifold whose point represents a state of the object on which measurements are performed. In general, a measurement process is described as a time evolution of a state of a compound system consisting of the object and the environment surrounding it. In order to discriminate differences in the environments, we introduce a parameter λ. We denote the parameter space by Λ. Then a state of the compound system is represented by a point in the product space Ω of Γ and Λ, i.e., $\Omega = \Gamma \times \Lambda$. We denote the projection of Ω onto Γ by π which is defined as $\pi((\xi, \lambda)) := \xi$ for $\forall (\xi, \lambda) \in \Omega$.

It should be remarked here that there is a subtlety in the interpretation of the environment. The environment $\lambda \in \Lambda$ does not always represent a state of only the measuring apparatus. It is more likely to be true that the object is not an isolated system and the environment contains a part of the object that is not handled by such a mathematical concept as an analytic manifold. The quantum field theory suggests this; an electron is not an isolated particle, but is surrounded by a cloud of photons.

Though the measurement process depends on which property is measured, by including this difference into the parameter λ it is possible to represent time evolutions of states under any measurement processes by a single flow. The flow is a set $\{\tau_s\}_{s \in \mathbf{R}}$ of mappings of Ω into Ω that is parametrized by a real number s and enjoys that $\tau_t \circ \tau_s = \tau_{t+s}$, $\tau_0 = 1_\Omega$. For the sake of convenience, by reparametrizing the time parameter s, we can assume that a measurement process starts at $s = 0$ and finishes at $s = 1$ without loss of generality.

For $\xi_1, \xi_2 \in \Gamma$ put

$$\mathcal{E}_{\xi_1, \xi_2} := \{\omega \in \Omega \mid \pi(\omega) = \xi_2, \text{ there exists } \omega_1 \in \Omega \text{ such that } \pi(\omega_1) = \xi_1 \text{ and } \omega_2 = \tau_1(\omega_1)\}.$$

$\mathcal{E}_{\xi_1, \xi_2}$ is an event in which the state of the object prepared in the state $\xi_1 \in \Gamma$ becomes $\xi_2 \in \Gamma$ after a measurement. Let $\mathcal{B}$ be a σ-algebra that contains $\mathcal{E}_{\xi_1, \xi_2}$ ($\forall \xi_1, \xi_2 \in \Gamma$). Then $(\Omega, \mathcal{B})$ is a measurable space.

In general, since various states of the compound system could be prepared prior to measurements, outcomes of the measurements become probabilistic. Let P be a probability measure on $(\Omega, \mathcal{B})$ that describes this probabilistic behavior of the prepared compound system.

We denote by $\mathcal{W}_{\xi_0}(\xi; P)$ an unnormalized "probability" that the state of the object that is in a state $\xi_0 \in \Gamma$ before a measurement becomes $\xi \in \Gamma$ after it, i.e.,

$$\mathcal{W}_{\xi_0}(\xi; P) := P(\mathcal{E}_{\xi_0, \xi}). \tag{4}$$

It is clear by this definition that

$$\mathcal{W}_{\xi_0}(\xi; P) \geq 0 \text{ for } \forall \xi_0, \xi \in \Gamma. \tag{5}$$

We can now express the first and second premises of our model of measurements.

Premise 1. There exists $\xi_0 \in \Gamma$ such that

$$\mathcal{W}_{\xi_0}(\xi_0; P) = 0. \tag{6}$$

Remark. This premise means that the object changes its state from ξ_0 to another after any measurement. It does not mean nonexistence of the state ξ_0.

Premise 2. $\mathcal{W}_{\xi_0}(\xi; P)$ is analytic for ξ_0 in a neighbourhood of ξ_0 in a local coordinate system $(\xi^1, \dots, \xi^m)$.

By these premises and (5), we can see that

Proposition 1.

$$\mathcal{W}_{\xi_0}(\xi; P) = \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 \mathcal{W}_{\xi_0}(\xi_0; P)}{\partial \xi^i \, \partial \xi^j} (\xi^i - \xi_0^i)(\xi^j - \xi_0^j) + O(d(\xi, \xi_0)^3),$$
$$\left(d(\xi, \xi_0) := \sqrt{\sum_{i=1}^{m} (\xi^i - \xi_0^i)^2} \longrightarrow 0\right).$$

Suppose that we obtain a value $a_1 \in \mathbf{R}$ as an outcome of a measurement of some property. There exists $\omega_0 \in \Omega$ such that $\pi(\omega_0) = \xi_0$ and this measurement process is represented by a curve $\{\tau_s(\omega_0) \mid 0 \leq s \leq 1\}$ in Ω. The state of the object after the measurement is $\pi(\tau_1(\omega_0))$. Since an ideal measurement is considered as a limit of a sequence of measurements such that the variation of the state of the object approaches zero, the variation of the object's state under an ideal measurement is represented by an infinitesimal variation v that is a tangent vector at ξ_0. In the local coordinate system $(\xi^1, \ldots, \xi^m)$, $v \in T_{\xi_0}\Gamma$ is written as

$$v = \sum_{i=1}^{m} v^i \left(\frac{\partial}{\partial \xi^i}\right)_{\xi_0}, \qquad v^i \in \mathbf{R}. \tag{7}$$

Define a symmetric tensor W_P covariant of order 2 by

$$W_P := \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 \mathcal{W}_{\xi_0}(\xi_0;P)}{\partial \xi^i \partial \xi^j}\, d\xi^i \otimes d\xi^j \tag{8}$$

The unnormalized "probability" $w_P(v)$ for an ideal measurement that produces an infinitesimal variation $v \in T_{\xi_0}\Gamma$ is expressed as

$$w_P(v) := W_P(v,\ v). \tag{9}$$

By (5) and proposition 1, we can see that

$$w_P(v) \geq 0 \qquad \text{for } \forall v \in T_{\xi_0}\Gamma \tag{10}$$

and W_P is positive semidefinite.

The third premise of our model is the following:

Premise 3. If an ideal measurement that causes an infinitesimal variation $v \in T_{\xi_0}\Gamma$ of the object's state gives a value a_1 as the outcome, then an ideal measurement that causes the infinitesimal variation $-v$ gives the same value a_1 as its outcome.

Moreover, if there exists an almost complex structure J on Γ, then an ideal measurement that causes the infinitesimal variation $Jv \in T_{\xi_0}\Gamma$ of the object's state gives the same value a_1 as its outcome, too.

Remarks. It is clear that the first part of the above premise is required for the invariance of measurement processes under time reversal $(s \mapsto -s)$.

The second part requires some symmetry of measurement processes. We do not yet appreciate its meaning fully, but the second part is necessary to deal with complex numbers. We try to find a candidate of this symmetry in the following. By the existence of the almost complex structure J, the dimension m of Γ must be even; there exists a natural number n such that $m = 2n$. Let $(x^1, \ldots, x^n, p^1, \ldots, p^n)$ be a local coordinate system such that

$$J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial p^i}, \quad J\frac{\partial}{\partial p^i} = -\frac{\partial}{\partial x^i}, \quad i = 1, \ldots, n. \tag{11}$$

Put $z^i := (x^i+\sqrt{-1}p^i)/2$ and $\bar{z}^i := (x^i - \sqrt{-1}p^i)/2$ $(i = 1, \ldots, n)$. Then by extending J to the complexification of $T_{\xi_0}\Gamma$ by linearity, we have

$$J\frac{\partial}{\partial z^i} = \sqrt{-1}\,\frac{\partial}{\partial z^i}, \quad J\frac{\partial}{\partial \bar{z}^i} = -\sqrt{-1}\,\frac{\partial}{\partial \bar{z}^i}, \quad i = 1, \ldots, n. \tag{12}$$

Let $H(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)$ be a Hamiltonian. The n Hamilton's canonical equations of motion become

$$\frac{dz^i}{dt} = -\sqrt{-1}\frac{\partial H}{\partial \bar{z}^i}, \quad i = 1, \ldots, n. \tag{13}$$

Consider a transformation $z^i \mapsto \zeta^i$ defined by

$$\zeta^i = -\sqrt{-1}z^i. \tag{14}$$

Then

$$\frac{\partial}{\partial \zeta^i} = \sqrt{-1}\frac{\partial}{\partial z^i} = J\frac{\partial}{\partial z^i},$$
$$\frac{\partial}{\partial \bar{\zeta}^i} = -\sqrt{-1}\frac{\partial}{\partial \bar{z}^i} = J\frac{\partial}{\partial \bar{z}^i}.$$

Under this transformation, Hamilton's canonical equation of motion becomes

$$\frac{d\zeta^i}{dt} = -\sqrt{-1}\frac{\partial H(\sqrt{-1}\zeta^1, \ldots)}{\partial \bar{\zeta}^i}. \tag{15}$$

If the Hamiltonian H depends on only $|z^1|^2, \ldots, |z^n|^2$, then the measurement process is invariant under this transformation.

By premise 3, if a measurement process whose outcome is a_1 causes a variation $v \in T_{\xi_0}\Gamma$ of the object's state, then an outcome of a measurement process that causes a variation $-v$ (or Jv) of the state is a_1. Therefore the unnormalized "probability" of the outcome being a_1 is given by

$$\rho_P(v) := w_P(v) + w_P(-v) \tag{16}$$
$$\left(\text{or } \rho_P(v) := w_P(v) + w_P(-v) + w_P(Jv) + w_P(-Jv)\right). \tag{17}$$

It is clear that

$$\rho_P(-v) = \rho_P(v). \tag{18}$$

For ρ_P defined by (17), it is also clear that

$$\rho_P(Jv) = \rho_P(v) \tag{19}$$

and (18) hold.

Hereafter we restrict our considerations to the case that an almost complex structure J on Γ exists, because the other case is easy to handle. We denote the complexification of $T_{\xi_0}\Gamma$ by $T_{\xi_0}\Gamma^{\mathrm{C}}$. The domain of ρ_P defined by (17) shall be extended to $T_{\xi_0}\Gamma^{\mathrm{C}}$ in the following. Firstly, we define a positive semidefinite tensor Φ_P covariant of order 2 on $T_{\xi_0}\Gamma$ by

$$\Phi_P(u,\ v) := 2W_P(u,\ v) + 2W_P(Ju,\ Jv) \qquad \text{for } \forall u,\ v \in T_{\xi_0}\Gamma. \tag{20}$$

Clearly, $\rho_P(v) = \Phi_P(v,\ v)$. A computation yields

$$\Phi_P(Ju,\ Jv) := \Phi_P(u,\ v). \tag{21}$$

We extend Φ_P and J to the complexification $T_{\xi_0}\Gamma^{\mathrm{C}}$ of $T_{\xi_0}\Gamma$ by linearity. We denote the extensions by the same symbols as before.

Proposition 2.

$$\Phi_P\left(\frac{\partial}{\partial z^i},\ \frac{\partial}{\partial z^j}\right) = \Phi_P\left(\frac{\partial}{\partial \bar{z}^i},\ \frac{\partial}{\partial \bar{z}^j}\right) = 0. \tag{22}$$

proof.

$$\begin{aligned}\Phi_P\left(\frac{\partial}{\partial z^i},\ \frac{\partial}{\partial z^j}\right) &= \Phi_P\left(J\frac{\partial}{\partial z^i},\ J\frac{\partial}{\partial z^j}\right) = \Phi_P\left(\sqrt{-1}\frac{\partial}{\partial z^i},\ \sqrt{-1}\frac{\partial}{\partial z^j}\right) = \\ &= \sqrt{-1}^2\ \Phi_P\left(\frac{\partial}{\partial z^i},\ \frac{\partial}{\partial z^j}\right) = -\Phi_P\left(\frac{\partial}{\partial z^i},\ \frac{\partial}{\partial z^j}\right).\end{aligned}$$

Hence $\Phi_P\left(\frac{\partial}{\partial z^i},\ \frac{\partial}{\partial z^j}\right) = 0$. In the same way, we can see that $\Phi_P\left(\frac{\partial}{\partial \bar{z}^i},\ \frac{\partial}{\partial \bar{z}^j}\right) = 0.$ □

A straightforward calculation yields

Proposition 3.

$$\begin{aligned}\Phi_P\left(\frac{\partial}{\partial \bar{z}^i},\ \frac{\partial}{\partial z^j}\right) = \quad &\Phi_P\left(\frac{\partial}{\partial x^i},\ \frac{\partial}{\partial x^j}\right) + \Phi_P\left(\frac{\partial}{\partial p^i},\ \frac{\partial}{\partial p^j}\right) \\ &+\sqrt{-1}\left\{\Phi_P\left(\frac{\partial}{\partial x^j},\ \frac{\partial}{\partial p^i}\right) - \Phi_P\left(\frac{\partial}{\partial x^i},\ \frac{\partial}{\partial p^j}\right)\right\}.\end{aligned}$$

We can define the unnormalized "probability" ρ_P^{C} of complex number version by

$$\rho_P^{\mathrm{C}}(u) := \Phi_P(\bar{u},\ u), \qquad u \in T_{\xi_0}\Gamma^{\mathrm{C}}. \tag{23}$$

We can show by a straightforward calculation

Proposition 4. For $\forall u = \Sigma_{i=1}^n c^i\frac{\partial}{\partial z^i} \in T_{\xi_0}\Gamma^{\mathrm{C}}$, put $v = \Sigma_{i=1}^n(\mathrm{Re}c^i\frac{\partial}{\partial x^i} + \mathrm{Im}c^i\frac{\partial}{\partial p^i})$. Then we have

$$\rho_P^{\mathrm{C}}(u) = 2\rho_P(v). \tag{24}$$

Proposition 5. For $\forall u \in T_{\xi_0}\Gamma^{\mathrm{C}}$,

$$\rho_P{}^{\mathrm{C}}(u) \geq 0. \tag{25}$$

proof. Clear by proposition 4. $\square$

Let $a_1, \ldots, a_n$ ($\in$ R) be possible outcomes of measurements of a property of the object and $\alpha_1, \ldots, \alpha_n \in T_{\xi_0}\Gamma^{\mathrm{C}}$ be the corresponding infinitesimal variations of states of the object, respectively. We say that a_i is a *possible outcome* of a measurement of a property if it is possible to prepare the object and the environment that gives the outcome α_i with certainty. This preparation is necessary to calibrate the measuring apparatus for the property. If it is impossible to prepare such states of the compound system that gives an outcome a_i with certainty, then the measuring apparatus is invalid.

Proposition 6. Infinitesimal variations $\alpha_1, \ldots, \alpha_n \in T_{\xi_0}\Gamma^{\mathrm{C}}$ of states of the object under measurements of a property are linearly independent in $T_{\xi_0}\Gamma^{\mathrm{C}}$.

proof. Suppose that $\alpha_1, \ldots, \alpha_n$ are linearly dependent. Then one of them, say α_i, is a linear combination of others, i.e.,

$$\alpha_i = \sum_{j \neq i} c_i \alpha_j, \tag{26}$$

where at least one of c_j's $(j \neq i)$ is not zero. Let c_l be the nonzero coefficient.

By the validity of the measuring apparatus, it is possible to prepare an ensemble, represented by a probability measure P_{a_l}, of compound systems such that the unnormalized "probability" $\rho_{P_{a_l}}{}^{\mathrm{C}}$ enjoys the following properties:

$$\rho_{P_{a_l}}{}^{\mathrm{C}}(\alpha_l) \neq 0, \quad \rho_{P_{a_l}}{}^{\mathrm{C}}(\alpha_j) = 0 \text{ for } \forall j \neq l. \tag{27}$$

By Schwartz's inequality

$$|\, \Phi_{P_{a_l}}(\bar{\alpha}_j, \ \alpha_k) \,|^2 \leq \Phi_{P_{a_l}}(\bar{\alpha}_j, \ \alpha_j) \Phi_{P_{a_l}}(\bar{\alpha}_k, \ \alpha_k). \tag{28}$$

Since $\Phi_{P_{a_l}}(\bar{\alpha}_j, \ \alpha_j) = \rho_{P_{a_l}}{}^{\mathrm{C}}(\alpha_j) = 0$ for $j \neq l$, we have $\Phi_{P_{al}}(\bar{\alpha}_j, \ \alpha_k) = 0$ for $j \neq l$.

$$\rho_{P_{a_l}}{}^{\mathrm{C}}(\alpha_i) = \sum_{j,k \neq i} \bar{c}_j c_k \Phi_{P_{a_l}}(\bar{\alpha}_j, \ \alpha_k) = |\, c_l \,|^2 \ \Phi_{P_{a_l}}(\bar{\alpha}_l, \ \alpha_l) \neq 0. \tag{29}$$

Since $i \neq l$, this contradicts definition of $\rho_{P_{a_l}}{}^{\mathrm{C}}$. Therefore $\alpha_1, \ldots, \alpha_n$ are linearly independent. $\square$

Let $\alpha_1, \ldots, \alpha_n \in T_{\xi_0}\Gamma^{\mathrm{C}}$ be infinitesimal variations of states of the object under measurements of some property. By proposition 6, $\alpha_1, \ldots, \alpha_n$ forms a basis of $T_{\xi_0}\Gamma^{\mathrm{C}}$. Let $\{\sigma_1, \ldots, \sigma_n\}$ be the dual basis, i.e.,

$$\sigma_i(a_j) = \delta_{ij}, \quad i, j = 1, \ldots, n, \tag{30}$$

where δ_{ij} is the Kronecker's delta symbol.

We define an inner product $\langle \cdot, \cdot \rangle$ on $T_{\xi_0}\Gamma^{\mathrm{C}}$ by

$$\langle a_i, v \rangle := \sigma_i(v), \quad v \in T_{\xi_0}\Gamma^{\mathrm{C}},$$
$$\langle u, v \rangle := \overline{\langle v, u \rangle}, \quad u, v \in T_{\xi_0}\Gamma^{\mathrm{C}}.$$

Since $\dim T_{\xi_0}\Gamma^{\mathrm{C}} = n < \infty$, $T_{\xi_0}\Gamma^{\mathrm{C}}$ is a Hilbert space, which we denote by $\mathcal{H}$.

Put

$$\varphi(v) := \frac{1}{\sum_{i=1}^{n} \rho_P{}^{\mathrm{C}}(a_i)} \rho_P{}^{\mathrm{C}}(v), \quad \forall v \in T_{\xi_0}\Gamma^{\mathrm{C}}. \tag{31}$$

Since $\sum_{i=1}^{n} \varphi(a_i) = 1$, $\varphi(a_i)$ is a probability of the outcome being a_i. By taking an appropriate set $\mathcal{P}$ of projections on $\mathcal{H}$, $(\mathcal{H}, \mathcal{P}, \varphi)$ becomes a QMPS. Thus in our model of measurements, a quantum-mechanical state φ represents an ensemble of states of the compound system represented by a probability measure P.

If a quantum-mechanical state φ is decomposed as

$$\varphi = c_1 \varphi_1 + c_2 \varphi_2, \tag{32}$$

where φ_1 and φ_2 are different quantum-mechanical states, $c_1, c_2 > 0$ and $c_1 + c_2 = 1$, then we say that φ is a quantum-mechanical ***mixed*** state. A quantum-mechanical ***pure*** state is one that is not a quantum-mechanical mixed state.

lemma. Let $\beta_1, \ldots, \beta_n$ be infinitesimal variations of states of the object under measurements of a property to which values $b_1, \ldots, b_n$ as outcomes correspond, respectively. The quantum-mechanical state φ_{b_1} that validates that the measuring apparatus can measure the value b_1 is a quantum-mechanical pure state

$$\varphi_{b_1}(\cdot) = \frac{1}{\langle \beta_1, \beta_1 \rangle} \langle \cdot, \beta_1 \rangle \langle \beta_1, \cdot \rangle. \tag{33}$$

proof. In our model, φb_1 is given by a corresponding ensemble represented by a probability measure P_{b_1}, i.e.,

$$\varphi_{b_1}(\cdot) = \frac{1}{\sum_i \rho_{P_{b_1}}{}^{\mathrm{C}}(a_i)} \Phi_{P_{b_1}}(\bar{\cdot}, \cdot). \tag{34}$$

Let $\gamma_1, \ldots, \gamma_n$ be a complete orthonormal system of $\mathcal{H}$ that gives the spectrum decomposition of the right-hand side of (34) as below.

$$\varphi_{b_1}(v) = \sum_{j=1}^{n} c_j \langle v, \gamma_i \rangle \langle \gamma_i, v \rangle, \qquad c_j \geq 0. \tag{35}$$

Since $\Sigma_i \varphi_{b_1}(\alpha_i) = 1$, $\Sigma_i c_i = 1$ holds. We normalize the β_j's and denote them by $\hat{\beta}_j$'s, respectively. If there are more than two nonvanishing c_j's, then

$$\varphi_{b_1}(\hat{\beta}_1) < \sum_{j=1}^{n} \langle \hat{\beta}_1, \gamma_i \rangle \langle \gamma_i, \hat{\beta}_1 \rangle = \langle \hat{\beta}_1, \hat{\beta}_1 \rangle = 1. \tag{36}$$

While, for a quantum-mechanical pure state ψ_{β_1} defined by

$$\psi_{\beta_1}(\cdot) := \langle \cdot, \hat{\beta}_1 \rangle \langle \hat{\beta}_1, \cdot \rangle, \tag{37}$$

$\Sigma_i \psi_{\beta_1}(\alpha_i) = 1$ and $\psi_{\beta_1}(\hat{\beta}_1) = 1$. Thus the probability of the outcome being b_1 for φ_{b_1} is less than that for ψ_{β_1}. This means that there are some events in which the outcomes are not b_1 for φ_{b_1}. This contradicts that the outcome b_1 is obtained with certainty for φ_{b_1}. Hence all c_i's vanish except one, say c_l. Then since $c_l = 1$, $\varphi_{b_1}(\cdot) = \langle \cdot, \gamma_l \rangle \langle \gamma_l, \cdot \rangle$. Since $\varphi_{b_1}(\hat{\beta}_1)$ must be unity, we have $\gamma_l = e^{i\theta} \beta_1$. Therefore $\varphi_{b_1}(\cdot) = \langle \cdot, \hat{\beta}_1 \rangle \langle \hat{\beta}_1, \cdot \rangle$. $\square$

Proposition 7. Infinite variations $\beta_1, \ldots, \beta_n$ under measurements of a property of the object form an orthogonal system in $\mathcal{H}$.

proof. Let φ_{b_i} be the quantum-mechanical state that validates the outcome b_i corresponding to β_i. $\varphi_{b_i}(\beta_j) = \langle \beta_i, \beta_i \rangle \delta_{ij}$ holds.

By the above lemma, φ_{b_i} is given as

$$\varphi_{b_i}(\cdot) = \frac{1}{\langle \beta_i, \beta_i \rangle} \langle \cdot, \beta_i \rangle \langle \beta_i, \cdot \rangle. \tag{38}$$

Therefore

$$\langle \beta_i, \beta_j \rangle = \langle \beta_i, \beta_i \rangle \delta_{ij}. \tag{39}$$

This completes the proof. $\square$

The following proposition is clear from proposition 7.

Proposition 8. Normalized infinite variations $\hat{\beta}_1, \ldots, \hat{\beta}_n$ under measurements of a property of the object form a complete orthonormal system in $\mathcal{H}$. Therefore they give a context $\underline{\beta}$ in $(\mathcal{H}, \mathcal{P})$.

As a consequence, we have shown the following theorem.

Theorem. Probabilistic behavior of outcomes of the ideal measurements for the model in this paper based on measure-theoretic probability theory with premises 1 to 3 is described by a QMPS.

References

1) Kochen, S. and Specker, E. P., (1967) *J. Math. Mech.* 17: 59

2) Beltrametti, E. G. and Cassinelli, G., (1981) *The Logic of Quantum Mechanics*, Addison-Wesley, Massacusetts.

3) Accardi, L., (1988) in Tarozzi, G. and van der Merwe, A., *The Nature of Quantum Paradoxes*, p. 257, Kluwer Academic Publishers.

4) Cufaro-Petroni, N., (1992) *Found. Phys.* 22: 1379.

5) Kolomogorov, A. N., (1933) *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer-Verlag, Berlin. See also, Halrmos, P. R., (1950)*Measure Theory*, Van Nostrand, Princeton. N. J.

6) Wick, G. C., Wightman, A. S. and Wigner, E. P., (1952) *Phys. Rev.* 88: 101.

7) Shewell, J. R., (1959) *Am. J. Phys.* 27: 16.

8) Ishigaki, T., (1991) *Found. Phys.* 21: 1089.

9) von Neumann, J., (1932) *Mathematische Grundlagen der Quantenmechnik*, Julius Springer-Verlag, Berlin [Japanese trnsl. : Misuzu-syobou, Tokyo, 1957].

10) Wiener, N. and Siegel, A., (1955) *Nuovo Cimeto Suppl.* 2: 982.

11) Bohm, D. and Hiley, B. J., (1993) *Undivided Universe - An ontological interpretation of quantum theory*, Rouyledge, London.