Representations of the Lie Algebra of $U(n, \mathbf{C})$ in Terms of Poisson Brackets

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Contents

Ⅰ. Introduction

It is well known that a certain set of matrices forms a Lie algebra in which the Lie products are defined by commutators of matrices(1). It is also well known that a certain set of functions on a symplectic manifold forms a Lie algebra in which the Lie products are defined by the Poisson brackets(2). Lie algebras given by the Poisson brackets were studied by the Norwegian mathematician Sophus Lie as a first step of study of noncommutative Lie algebras(3). The Lie algebra of all differentiable functons whose Lie products are the Poisson brackts is inifinite-dimensional. The Lie algebra of finite dimensional matrices whose Lie products are commutators is finite-dimensional. An infinite-dimensional linear space has a finite-dimensional linear subspace. Then, is it possible to embed the finite-dimensional Lie algebra of matrices into the Lie algebra of differentiable functions as a subalgebra?

This question is worth asking in particular in the case of $\mathbf{u}(n)$. On the one hand, the Lie algebra $\mathbf{u}(n)$ of the classical Lie group $U(n, \mathbf{C})$, i.e., n-dimensional unitary group, is a framework of quantum-mechanical probability theory(4). On the other hand, in classical mechanics, physical quantities are represented by functions defined on a classical phase space, and they constitute a Lie algebra by Poisson brackets. It may be possible to get some insight on quantum-mechanical probability from a viewpoint of the

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realism underlying the classical mechanics, if there is some relation between them.

In this paper, one of the answers to the question is presented. The answer is that there exists faithful representations of $\mathbf{u}(n)$ as subalgebras of Lie algebars of differentiable functions on a $2n$-dimensional symplectic space.

Ⅱ. A base of the Lie algebra $\mathbf{u}(n)$

In this section, we introduce some symbols for the elements of $\mathbf{u}(n)$, and define a base of $\mathbf{u}(n)$.

We denote the set of all $n \times n$ matrices with complex entries by $M(n, \mathbf{C})$. Let $E_{ij} \in M(n, \mathbf{C})$ be a $n \times n$ matrix such that only the (i, j) -entry is unity and other entries vanish, i.e.,

$$(E_{ij})_{kl} := \delta_{ik} \delta_{jl}, \quad (1)$$

where δ_{ij} is a Kronecker's delta.

The set $\{E_{ij} \mid i, j = 1, ..., n\}$ constitutes a base of $\mathbf{gl}(n, \mathbf{C})$ which is the Lie algebra of the general linear group $GL(n, \mathbf{C})$.

For products of them, we have the following relation,

$$E_{ij} E_{kl} = \delta_{jk} E_{il}, \quad i, j, k, l = 1, ..., n. \quad (2)$$

This is because

$$\begin{aligned}(E_{ij}E_{kl})_{ab} &= \sum_{c=1}^{n} (E_{ij})_{ac} (E_{kl})_{cb} \\ &= \sum_{c=1}^{n} \delta_{ia}\delta_{jc}\delta_{kc}\delta_{lb} \\ &= \delta_{ia}\delta_{kj}\delta_{lb} \\ &= \delta_{kj}\delta_{ia}\delta_{lb} \\ &= \delta_{kj} (E_{il})_{ab}.\end{aligned}$$

Using this relation, we obtain

$$\begin{aligned}[E_{ij}, E_{kl}] &:= E_{ij} E_{kl} - E_{kl} E_{ij} \\ &= \delta_{jk} E_{il} - \delta_{li} E_{kj}. \quad (3)\end{aligned}$$

The unitary group $U(n, \mathbf{C})$ is defined as a subgroup of $GL(n, \mathbf{C})$ whose element $U \in M(n, \mathbf{C})$ satisfies that

$$U^{\dagger} = U, \quad (4)$$

where the symbol † means a composite operation of transposition and complex conjugation. Since the exponential map is an injection of the Lie algebra into its corresponding Lie group[(1)],

$$\mathbf{u}(n) = \{A \in M(n, \mathbf{C}) \mid A^{\dagger} = -A\}. \quad (5)$$

Because $\{E_{ij}\}$ is a base of $\mathbf{gl}(n, \mathbf{C})$, it is clear that

$$\mathbf{u}(n) = \mathrm{span}\left\{\sqrt{-1}E_{ii}, -E_{ij} + E_{ji}, \sqrt{-1}(E_{ij} + E_{ji}) \mid i, j = 1, \ldots, n\right\}, \quad (6)$$

where “ span ” means taking all linear combinations of the set with real coefficients.

We introduce the following symbols.

$$T_i := E_{ii}, \tag{7}$$
$$X_{ij} := E_{ij} + E_{ji}, \tag{8}$$
$$Y_{ij} := \sqrt{-1}(E_{ij} - E_{ji}). \tag{9}$$

In addition to the above, we also introduce the following:

$$JT_i := \sqrt{-1}T_i = \sqrt{-1}E_{ii}, \tag{10}$$
$$JX_{ij} := \sqrt{-1}X_{ij} = \sqrt{-1}(E_{ij} + E_{ji}), \tag{11}$$
$$JY_{ij} := \sqrt{-1}Y_{ij} = -E_{ij} + E_{ji}. \tag{12}$$

Note that $\{JT_i, JX_{ij}, JY_{ij}\}$ is a base of $\mathbf{u}(n)$, but $\{T_i, X_{ij}, Y_{ij}\}$ is not. Since

$$JX_{ji} = JX_{ij}, \tag{13}$$

$$JY_{ji} = -JY_{ij}, \tag{14}$$

the dimension of $\mathbf{u}(n)$ is n^2.

The task that remains in this section is to find the structure constants of $\mathbf{u}(n)$ with respect to the base $\{JT_i, JX_{ij}, JY_{ij}\}$. A product of two elements $A, B \in \mathbf{u}(n)$ in a Lie algebra is given by their commutator $[A, B]$ defined by $[A, B] := AB - BA$.

$$\begin{aligned}
[JT_i, JX_{jk}] &= -[E_{ii}, E_{jk} + E_{kj}] \\
&= -[E_{ii}, E_{jk}] - [E_{ii}, E_{kj}] \\
&= -\delta_{ij}E_{ik} + \delta_{ik}E_{ji} - \delta_{ik}E_{ij} + \delta_{ji}E_{ki} \\
&= \delta_{ij}(-E_{ik} + E_{ki}) + \delta_{ik}(E_{ji} - E_{ij}) \\
&= \delta_{ij}\sqrt{-1}Y_{ik} + \delta_{ik}\sqrt{-1}Y_{ij} \\
&= \delta_{ij}JY_{ik} + \delta_{ik}JY_{ij} \\
&= \delta_{ij}JY_{jk} + \delta_{ik}JY_{kj} \\
&= \delta_{ij}JY_{jk} - \delta_{ik}JY_{jk} \\
&= (\delta_{ij} - \delta_{ik})JY_{jk}.
\end{aligned}$$

Hence

$$[JT_i, JX_{jk}] = (\delta_{ij} - \delta_{ik})\, JY_{jk}. \tag{15}$$

$$\begin{aligned}
[JT_i, JY_{jk}] &= \sqrt{-1}[E_{ii}, -E_{jk} + E_{kj}] \\
&= -\sqrt{-1}[E_{ii}, E_{jk}] + \sqrt{-1}[E_{ii}, E_{kj}] \\
&= -\sqrt{-1}\delta_{ij}E_{ik} + \sqrt{-1}\delta_{ik}E_{ji} + \sqrt{-1}\delta_{ik}E_{ij} - \sqrt{-1}\delta_{ji}E_{ki} \\
&= \sqrt{-1}\delta_{ij}(-E_{ik} - E_{ki}) + \sqrt{-1}\delta_{ik}(E_{ji} + E_{ij}) \\
&= -\sqrt{-1}\delta_{ij}X_{ik} + \sqrt{-1}\delta_{ik}X_{ij} \\
&= -\delta_{ij}JX_{ik} + \delta_{ik}JX_{ij} \\
&= -\delta_{ij}JX_{jk} + \delta_{ik}JX_{kj} \\
&= -\delta_{ij}JX_{jk} + \delta_{ik}JX_{jk} \\
&= -(\delta_{ij} - \delta_{ik})JX_{jk}.
\end{aligned}$$

Therefore

$$[JT_i, JY_{jk}] = -(\delta_{ij} - \delta_{ik})\, JX_{jk}. \tag{16}$$

$$\begin{aligned}[JX_{ij}, JY_{kl}] &= \sqrt{-1}[E_{ij}+E_{ji}, -E_{kl}+E_{lk}] \\ &= -\sqrt{-1}[E_{ij}, E_{kl}] + \sqrt{-1}[E_{ij}, E_{lk}] - \sqrt{-1}[E_{ji}, E_{kl}] + \sqrt{-1}[E_{ji}, E_{lk}] \\ &= -\sqrt{-1}(\delta_{jk}E_{il} - \delta_{li}E_{kj}) + \sqrt{-1}(\delta_{jl}E_{ik} - \delta_{ki}E_{lj}) \\ &\quad -\sqrt{-1}(\delta_{ik}E_{jl} - \delta_{lj}E_{ki}) + \sqrt{-1}(\delta_{il}E_{jk} - \delta_{kj}E_{li}) \\ &= -\sqrt{-1}\delta_{jk}(E_{il}+E_{li}) + \sqrt{-1}\delta_{li}(E_{kj}+E_{jk}) \\ &\quad +\sqrt{-1}\delta_{jl}(E_{ik}+E_{ki}) - \sqrt{-1}\delta_{ik}(E_{jl}+E_{lj}) \\ &= -\delta_{jk}JX_{il} + \delta_{li}JX_{kj} + \delta_{jl}JX_{ik} - \delta_{ik}JX_{jl}.\end{aligned}$$

Thus we obtained

$$[JX_{ij}, JY_{kl}] = \delta_{il}JX_{jk} - \delta_{jk}JX_{il} + \delta_{jl}JX_{ik} - \delta_{ik}JX_{jl}. \tag{17}$$

In particular, if $(k, l) = (i, j)$, $i \neq j$, then

$$\begin{aligned}[JX_{ij}, JY_{ij}] &= \delta_{jj}JX_{ii} - \delta_{ii}JX_{jj} \\ &= 2\sqrt{-1}(E_{ii} - E_{jj}) \\ &= 2\sqrt{-1}(T_i - T_j) \\ &= 2JT_i - 2JT_j.\end{aligned}$$

We obtained

$$[JX_{ij}, JY_{ij}] = 2JT_i - 2JT_j. \tag{18}$$

$$\begin{aligned}[JX_{ij}, JX_{kl}] &= -[E_{ij}+E_{ji}, E_{kl}+E_{lk}] \\ &= -[E_{ij}, E_{kl}] - [E_{ij}, E_{lk}] - [E_{ji}, E_{kl}] - [E_{ji}, E_{lk}] \\ &= -(\delta_{jk}E_{il} - \delta_{li}E_{kj}) - (\delta_{jl}E_{ik} - \delta_{ki}E_{lj}) - (\delta_{ik}E_{jl} - \delta_{lj}E_{ki}) - (\delta_{il}E_{jk} - \delta_{kj}E_{li}) \\ &= -\delta_{jk}(E_{il} - E_{li}) + \delta_{li}(E_{kj} - E_{jk}) - \delta_{jl}(E_{ik} - E_{ki}) - \delta_{ik}(E_{jl} - E_{lj}) \\ &= \delta_{jk}JY_{il} + \delta_{li}JY_{jk} + \delta_{jl}JY_{ik} + \delta_{ik}JY_{jl}.\end{aligned}$$

Therefore

$$[JX_{ij}, JX_{kl}] = \delta_{jk}JY_{il} + \delta_{li}JY_{jk} + \delta_{jl}JY_{ik} + \delta_{ik}JY_{jl}. \tag{19}$$

$$\begin{aligned}[JY_{ij}, JY_{kl}] &= [-E_{ij}+E_{ji}, -E_{kl}+E_{lk}] \\ &= [E_{ij}, E_{kl}] - [E_{ij}, E_{lk}] - [E_{ji}, E_{kl}] + [E_{ji}, E_{lk}] \\ &= (\delta_{jk}E_{il} - \delta_{li}E_{kj}) - (\delta_{jl}E_{ik} - \delta_{ki}E_{lj}) - (\delta_{ik}E_{jl} - \delta_{lj}E_{ki}) + (\delta_{il}E_{jk} - \delta_{kj}E_{li}) \\ &= \delta_{jk}(E_{il} - E_{li}) - \delta_{li}(E_{kj} - E_{jk}) - \delta_{jl}(E_{ik} - E_{ki}) - \delta_{ik}(E_{jl} - E_{lj}) \\ &= -\delta_{jk}JY_{il} - \delta_{li}JY_{jk} + \delta_{jl}JY_{ik} + \delta_{ik}JY_{jl}.\end{aligned}$$

Therefore

$$[JY_{ij}, JY_{kl}] = -\delta_{jk}JY_{il} - \delta_{li}JY_{jk} + \delta_{jl}JY_{ik} + \delta_{ik}JY_{jl}. \tag{20}$$

The structure constants are shown in the equations (15), (16), (17), (18), (19) and (20).

Ⅲ. A representation of the Lie algebra $\mathbf{u}(n)$ as a subalgebra of the Lie algebra of functions

Let Λ be a product space $\mathbf{R}^{2n}$ of $2n$ sets of real numbers. Let $(\lambda^1, ..., \lambda^n, \mu_1, ..., \mu_n)$ be a coordinate system of Λ. Then a symplectic structure ω on Λ is defined in a standard way(2), i.e.,

$$\omega := \sum_{i=1}^{n} d\mu_i \wedge d\lambda^i. \tag{21}$$

In this way, Λ becomes a $2n$-dimensional symplectic manifold.

For a complex-valued function f on Λ, a Hamiltonian vector field $\xi_f \in T\Lambda$ is defined by

$$\omega(\xi_f, \cdot) = -df. \tag{22}$$

In the present case,

$$\xi_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \mu_i} \frac{\partial}{\partial \lambda^i} - \frac{\partial f}{\partial \lambda^i} \frac{\partial}{\partial \mu_i} \right). \tag{23}$$

The Poisson bracket $\{f, g\}_\Lambda$ of two functions f and g is defined as

$$\{f, g\}_\Lambda := \xi_f \cdot g. \tag{24}$$

From eq. (23), we obtain

$$\{f, g\}_\Lambda = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \mu_i} \frac{\partial g}{\partial \lambda^i} - \frac{\partial f}{\partial \lambda^i} \frac{\partial g}{\partial \mu_i} \right). \tag{25}$$

A set of functions on Λ generates a Lie algebra whose Lie products are Poisson brackets, because Poisson brackets satisfy the Lie conditions, i.e.,

$$\{f, f\}_\Lambda = 0 \tag{26}$$

and the Jacobi identity

$$\{\{f, g\}_\Lambda, h\}_\Lambda + \{\{g, h\}_\Lambda, f\}_\Lambda + \{\{h, f\}_\Lambda, g\}_\Lambda = 0 \tag{27}$$

for functions f, g and h on Λ. Since the restriction of ω to an open subset Δ of Λ is a symplectic structure on Δ, a set of functions on Δ generates a Lie algebra. Generally, the Lie algebra of functions may be infinite-dimensional. Since the Lie algebra $\mathbf{u}(n)$ is of finite dimension, it can be expected that $\mathbf{u}(n)$ is a Lie subalgebra of an infinite-dimensional Lie algebra of some functions on some open subset Δ of Λ.

Let ε be a real number. We define an open set $\Delta_\varepsilon \subset \Lambda$ by

$$\Delta_\varepsilon := \{\eta \in \Lambda \mid 0 < \varepsilon + \lambda^i, i = 1, ..., n\}. \tag{28}$$

The main results of this paper is the following

Theorem. A Lie algebra generated by real-valued functions t_i, x_{ij}, y_{ij} *on* Δ_ε, $i, j = 1, ..., n$, $i < j$ defined in the following with Lie products defined by the Poisson brackets is isomorphic to $\mathbf{u}(n)$.

$$t_i := \lambda^i, \mathrm{i} = 1, ..., n. \tag{29}$$

$$x_{ij} := f_{ij}(\lambda)\cos(\mu_i - \mu_j + \phi_{ij}) = 2\sqrt{(\epsilon+\lambda^i)(\epsilon+\lambda^j)}\cos(\mu_i - \mu_j + \phi_{ij}), \tag{30}$$
$$y_{ij} := f_{ij}(\lambda)\sin(\mu_i - \mu_j + \phi_{ij}) = 2\sqrt{(\epsilon+\lambda^i)(\epsilon+\lambda^j)}\sin(\mu_i - \mu_j + \phi_{ij}), \tag{31}$$

where

$$\phi_{ij} := \theta_i - \theta_j,\ 1 \leq i < j \leq n,\ \theta_i \in \mathbf{R}. \tag{32}$$

Proof. It suffices to show that t_i, x_{ij}, y_{ij} correspond to JT_i, JX_{ij}, JY_{ij}, respectively, since the latter constitute a base of $\mathbf{u}(n)$.

For $1 \leq i, j, k \leq n, j < k$,

$$\begin{aligned}
\{t_i, x_{jk}\}_\Lambda &= \{\lambda^i, x_{jk}\}_\Lambda = -\frac{\partial x_{jk}}{\partial \mu_i} \\
&= -f_{jk}(\lambda)\frac{\partial}{\partial \mu_i}\cos(\mu_j - \mu_k + \phi_{jk}) \\
&= f_{jk}(\lambda)\sin(\mu_j - \mu_k + \phi_{jk})(\delta_{ji} - \delta_{ik}) \\
&= -y_{jk}(\delta_{ij} - \delta_{ik}) \\
&= (\delta_{ij} - \delta_{ik})y_{jk}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\{t_i, y_{jk}\}_\Lambda &= \{\lambda^i, y_{jk}\}_\Lambda = -\frac{\partial y_{jk}}{\partial \mu_i} \\
&= -f_{jk}(\lambda)\frac{\partial}{\partial \mu_i}\sin(\mu_j - \mu_k + \phi_{jk}) \\
&= -f_{jk}(\lambda)\cos(\mu_j - \mu_k + \phi_{jk})(\delta_{ji} - \delta_{ik}) \\
&= -(\delta_{ij} - \delta_{ik})x_{jk}.
\end{aligned}$$

For $1 \leq i, j, k, l \leq n, i < j, k < l$,

$$\begin{aligned}
\{y_{ij}, y_{kl}\}_\Lambda &= \{f_{ij}(\lambda)\sin(\mu_i - \mu_j + \phi_{ij}), f_{kl}(\lambda)\sin(\mu_k - \mu_l + \phi_{kl})\}_\Lambda \\
&= \{f_{ij}(\lambda), f_{kl}(\lambda)\sin(\mu_k - \mu_l + \phi_{kl})\}_\Lambda \sin(\mu_i - \mu_j + \phi_{ij}) \\
&\quad + f_{ij}(\lambda)\{\sin(\mu_i - \mu_j + \phi_{ij}), f_{kl}(\lambda)\sin(\mu_k - \mu_l + \phi_{kl})\}_\Lambda \\
&= \{f_{ij}(\lambda), \sin(\mu_k - \mu_l + \phi_{kl})\}_\Lambda f_{kl}(\lambda)\sin(\mu_i - \mu_j + \phi_{ij}) \\
&\quad + f_{ij}(\lambda)\{\sin(\mu_i - \mu_j + \phi_{ij}), f_{kl}(\lambda)\}_\Lambda \sin(\mu_k - \mu_l + \phi_{kl}) \\
&= \left(-\frac{\partial f_{ij}(\lambda)}{\partial \lambda^l}\frac{\partial \sin(\mu_k - \mu_l + \phi_{kl})}{\partial \mu_l} - \frac{\partial f_{ij}(\lambda)}{\partial \lambda^k}\frac{\partial \sin(\mu_k - \mu_l + \phi_{kl})}{\partial \mu_k}\right) \\
&\quad \times f_{kl}(\lambda)\sin(\mu_i - \mu_j + \phi_{ij}) \\
&\quad + \left(\frac{\partial \sin(\mu_i - \mu_j + \phi_{ij})}{\partial \mu_i}\frac{\partial f_{kl}(\lambda)}{\partial \lambda^i} + \frac{\partial \sin(\mu_i - \mu_j + \phi_{ij})}{\partial \mu_j}\frac{\partial f_{kl}(\lambda)}{\partial \lambda^j}\right) \\
&\quad \times f_{ij}(\lambda)\sin(\mu_k - \mu_l + \phi_{kl}) \\
&= \left(\frac{\partial f_{ij}(\lambda)}{\partial \lambda^l}\cos(\mu_k - \mu_l + \phi_{kl}) - \frac{\partial f_{ij}(\lambda)}{\partial \lambda^k}\cos(\mu_k - \mu_l + \phi_{kl})\right) \\
&\quad \times f_{kl}(\lambda)\sin(\mu_i - \mu_j + \phi_{ij}) \\
&\quad + \left(\cos(\mu_i - \mu_j + \phi_{ij})\frac{\partial f_{kl}(\lambda)}{\partial \lambda^i} - \cos(\mu_i - \mu_j + \phi_{ij})\frac{\partial f_{kl}(\lambda)}{\partial \lambda^j}\right) \\
&\quad \times f_{ij}(\lambda)\sin(\mu_k - \mu_l + \phi_{kl}) \\
&= \left(\frac{\partial f_{ij}(\lambda)}{\partial \lambda^l} - \frac{\partial f_{ij}(\lambda)}{\partial \lambda^k}\right) f_{kl}(\lambda)\cos(\mu_k - \mu_l + \phi_{kl})\sin(\mu_i - \mu_j + \phi_{ij}) \\
&\quad + \left(\frac{\partial f_{kl}(\lambda)}{\partial \lambda^i} - \frac{\partial f_{kl}(\lambda)}{\partial \lambda^j}\right) f_{ij}(\lambda)\cos(\mu_i - \mu_j + \phi_{ij})\sin(\mu_k - \mu_l + \phi_{kl}).
\end{aligned}$$

Since

we have

$$\frac{\partial f_{ij}}{\partial \lambda^i} = \frac{\sqrt{\epsilon+\lambda^j}}{\sqrt{\epsilon+\lambda^i}}, \tag{33}$$

$$\frac{\partial f_{ij}}{\partial \lambda^j} = \frac{\sqrt{\epsilon+\lambda^i}}{\sqrt{\epsilon+\lambda^j}}, \tag{34}$$

$$\frac{\partial f_{ij}}{\partial \lambda^i} f_{ik} = 2\sqrt{\epsilon+\lambda^j}\sqrt{\epsilon+\lambda^k} = f_{jk}, \quad 1 \le i, j, k \le n. \tag{35}$$

By using this,

$$\begin{aligned}
\{y_{ij}, y_{kl}\}_\Lambda &= \left(\frac{\partial f_{ij}(\lambda)}{\partial \lambda^l}(\delta_{il}+\delta_{jl}) - \frac{\partial f_{ij}(\lambda)}{\partial \lambda^k}(\delta_{ik}+\delta_{jk})\right) f_{kl}(\lambda) \\
&\quad \times \cos(\mu_k - \mu_l + \phi_{kl}) \sin(\mu_i - \mu_j + \phi_{ij}) \\
&\quad + \left(\frac{\partial f_{kl}(\lambda)}{\partial \lambda^i}(\delta_{ki}+\delta_{li}) - \frac{\partial f_{kl}(\lambda)}{\partial \lambda^j}(\delta_{kj}+\delta_{lj})\right) f_{ij}(\lambda) \\
&\quad \times \cos(\mu_i - \mu_j + \phi_{ij}) \sin(\mu_k - \mu_l + \phi_{kl}) \\
&= \left((\frac{\partial f_{lj}(\lambda)}{\partial \lambda^l}\delta_{il} + \frac{\partial f_{il}(\lambda)}{\partial \lambda^l}\delta_{jl}) - (\frac{\partial f_{kj}(\lambda)}{\partial \lambda^k}\delta_{ik} + \frac{\partial f_{ik}(\lambda)}{\partial \lambda^k}\delta_{jk}) \right) \\
&\quad \times f_{kl}(\lambda) \cos(\mu_k - \mu_l + \phi_{kl}) \sin(\mu_i - \mu_j + \phi_{ij}) \\
&\quad + \left((\frac{\partial f_{il}(\lambda)}{\partial \lambda^i}\delta_{ki} + \frac{\partial f_{ki}(\lambda)}{\partial \lambda^i}\delta_{li}) - (\frac{\partial f_{jl}(\lambda)}{\partial \lambda^j}\delta_{kj} + \frac{\partial f_{kj}(\lambda)}{\partial \lambda^j}\delta_{lj}) \right) \\
&\quad \times f_{ij}(\lambda) \cos(\mu_i - \mu_j + \phi_{ij}) \sin(\mu_k - \mu_l + \phi_{kl}) \\
&= [(f_{kj}(\lambda)\delta_{il} + f_{ik}(\lambda)\delta_{jl}) - (f_{lj}(\lambda)\delta_{ik} + f_{il}(\lambda)\delta_{jk})] \\
&\quad \times \cos(\mu_k - \mu_l + \phi_{kl}) \sin(\mu_i - \mu_j + \phi_{ij}) \\
&\quad + [(f_{jl}(\lambda)\delta_{ki} + f_{kj}(\lambda)\delta_{li}) - (f_{il}(\lambda)\delta_{kj} + f_{ki}(\lambda)\delta_{lj})] \\
&\quad \times \cos(\mu_i - \mu_j + \phi_{ij}) \sin(\mu_k - \mu_l + \phi_{kl}) \\
&= f_{kj}(\lambda)\delta_{il}[\cos(\mu_k - \mu_l + \phi_{kl}) \sin(\mu_i - \mu_j + \phi_{ij}) \\
&\quad + \cos(\mu_i - \mu_j + \phi_{ij}) \sin(\mu_k - \mu_l + \phi_{kl})] \\
&\quad + f_{ik}(\lambda)\delta_{jl}[\cos(\mu_k - \mu_l + \phi_{kl}) \sin(\mu_i - \mu_j + \phi_{ij}) \\
&\quad - \cos(\mu_i - \mu_j + \phi_{ij}) \sin(\mu_k - \mu_l + \phi_{kl})] \\
&\quad - f_{lj}(\lambda)\delta_{ik}[\cos(\mu_k - \mu_l + \phi_{kl}) \sin(\mu_i - \mu_j + \phi_{ij}) \\
&\quad - \cos(\mu_i - \mu_j + \phi_{ij}) \sin(\mu_k - \mu_l + \phi_{kl})] \\
&\quad - f_{il}(\lambda)\delta_{jk}[\cos(\mu_k - \mu_l + \phi_{kl}) \sin(\mu_i - \mu_j + \phi_{ij}) \\
&\quad + \cos(\mu_i - \mu_j + \phi_{ij}) \sin(\mu_k - \mu_l + \phi_{kl})] \\
&= f_{kj}(\lambda)\delta_{il} \sin(\mu_k - \mu_l + \phi_{kl} + \mu_i - \mu_j + \phi_{ij}) \\
&\quad + f_{ik}(\lambda)\delta_{jl} \sin(\mu_i - \mu_j + \phi_{ij} - \mu_k + \mu_l - \phi_{kl}) \\
&\quad - f_{lj}(\lambda)\delta_{ik} \sin(\mu_i - \mu_j + \phi_{ij} - \mu_k + \mu_l - \phi_{kl}) \\
&\quad - f_{il}(\lambda)\delta_{jk} \sin(\mu_k - \mu_l + \phi_{kl} + \mu_i - \mu_j + \phi_{ij}) \\
&= f_{kj}(\lambda)\delta_{il} \sin(\mu_k - \mu_j + \phi_{kl} + \phi_{ij}) \\
&\quad + f_{ik}(\lambda)\delta_{jl} \sin(\mu_i - \mu_k + \phi_{ij} - \phi_{kl}) \\
&\quad - f_{lj}(\lambda)\delta_{ik} \sin(\mu_l - \mu_j + \phi_{ij} - \phi_{kl}) \\
&\quad - f_{il}(\lambda)\delta_{jk} \sin(\mu_i - \mu_l + \phi_{kl} + \phi_{ij}).
\end{aligned}$$

Here, for convenience, we define

$$x_{ji} := x_{ij}, \tag{36}$$

$$y_{ji} := -y_{ij}, \tag{37}$$

for $1 \le i < j \le n$. These notations enable us to write the result in a general expression. Remembering eq. (32), we have

$$\begin{aligned}\{y_{ij}, y_{kl}\}_\Lambda &= \delta_{li} f_{jk}(\lambda)\sin(\mu_k - \mu_j + \phi_{ki} + \phi_{ij}) + \delta_{jl} f_{ik}(\lambda)\sin(\mu_i - \mu_k + \phi_{ij} - \phi_{kj}) \\ &\quad - \delta_{ik} f_{jl}(\lambda)\sin(\mu_l - \mu_j + \phi_{ij} - \phi_{il}) - \delta_{jk} f_{il}(\lambda)\sin(\mu_i - \mu_l + \phi_{jl} + \phi_{ij}) \\ &= \delta_{li} f_{jk}(\lambda)\sin(\mu_k - \mu_j + \phi_{kj}) + \delta_{jl} f_{ik}(\lambda)\sin(\mu_i - \mu_k + \phi_{ik}) \\ &\quad - \delta_{ik} f_{jl}(\lambda)\sin(\mu_l - \mu_j + \phi_{lj}) - \delta_{jk} f_{il}(\lambda)\sin(\mu_i - \mu_l + \phi_{il}) \\ &= -\delta_{li} f_{jk}(\lambda)\sin(\mu_j - \mu_k + \phi_{jk}) + \delta_{jl} f_{ik}(\lambda)\sin(\mu_i - \mu_k + \phi_{ik}) \\ &\quad + \delta_{ik} f_{jl}(\lambda)\sin(\mu_j - \mu_l + \phi_{jl}) - \delta_{jk} f_{il}(\lambda)\sin(\mu_i - \mu_l + \phi_{il}) \\ &= -\lambda_{li} y_{jk} + \lambda_{jl} y_{ik} + \lambda_{ik} y_{jl} - \lambda_{jk} y_{il}.\end{aligned}$$

Therefore we obtained

$$\{y_{ij}, y_{kl}\}_\Lambda = -\delta_{li} y_{jk} + \delta_{jl} y_{ik} + \delta_{ik} y_{jl} - \delta_{jk} y_{il},\ 1 < i, j, k, l < n. \tag{38}$$

We note that the following equations hold.

$$y_{ij}(\lambda, ..., \mu_j \mp \pi/2, ...) = \pm x_{ij}(\lambda, ..., \mu_j, ...),\ 1 \leq i < j \leq n, \tag{39}$$

$$y_{ij}(\lambda, ..., \mu_i \mp \pi/2, ...) = \mp x_{ij}(\lambda, ..., \mu_i, ...),\ 1 \leq i < j \leq n. \tag{40}$$

For the sake of simplicity, we introduce some notations of these equations as follows:

$$y_{ij}|_{\mu_j \mp \pi/2} = \pm x_{ij}|_{\mu_j}, \tag{41}$$

$$y_{ij}|_{\mu_i \mp \pi/2} = \mp x_{ij}|_{\mu_i},\ 1 \leq i < j \leq n\ . \tag{42}$$

First, suppose that $1 \leq i, j, k, l \leq n$, $i < j$, $k < l$. If i, j, k, l are different, then

$$\begin{aligned}\{x_{ij}, x_{kl}\}_\Lambda &= \{y_{ij}|_{\mu_j - \pi/2,\ \mu_l - \pi/2}, y_{kl}|_{\mu_j - \pi/2,\ \mu_l - \pi/2}\}_\Lambda \\ &= (-\delta_{li} y_{jk} + \delta_{jl} y_{ik} + \delta_{ik} y_{jl} - \delta_{jk} y_{il})|_{\mu_j - \pi/2,\ \mu_l - \pi/2} \\ &= 0 \\ &= \delta_{jk} y_{il} + \delta_{li} y_{jk} + \delta_{jl} y_{ik} + \delta_{ik} y_{jl}.\end{aligned}$$

If $k = j$, $i \neq l$, then

$$\begin{aligned}\{x_{ij}, x_{jl}\}_\Lambda &= -\{y_{ij}|_{\mu_j - \pi/2}, y_{jl}|_{\mu_j - \pi/2}\}_\Lambda \\ &= -(-\delta_{li} y_{jj} + \delta_{jl} y_{ij} + \delta_{ij} y_{jl} - \delta_{jj} y_{il})|_{\mu_j - \pi/2} \\ &= -(-\delta_{jj} y_{il})\ |_{\mu_j - \pi/2} \\ &= \delta_{jj} y_{il}\ |_{\mu_j} \\ &= \delta_{jk} y_{il} + \delta_{li} y_{jk} + \delta_{jl} y_{ik} + \delta_{ik} y_{jl}.\end{aligned}$$

If $l = j$, $i \neq k$, then

$$\begin{aligned}\{x_{ij}, x_{kj}\}_\Lambda &= \{y_{ij}|_{\mu_j - \pi/2}, y_{kj}|_{\mu_j - \pi/2}\}_\Lambda \\ &= (-\delta_{ji} y_{jk} + \delta_{jj} y_{ik} + \delta_{ik} y_{jj} - \delta_{jk} y_{ij})|_{\mu_j - \pi/2} \\ &= (\delta_{jj} y_{ik})|_{\mu_j - \pi/2} \\ &= \delta_{jl} y_{ik}|_{\mu_j} \\ &= \delta_{jk}\ y_{il} + \delta_{li} y_{jk} + \delta_{jl} y_{ik} + \delta_{ik} y_{jl}.\end{aligned}$$

Therefore we see that

$$\{x_{ij}, x_{kj}\}_\Lambda = \delta_{jk}\, y_{il} + \delta_{li}\, y_{jk} + \delta_{jl}\, y_{ik} + \delta_{ik}\, y_{jl} \tag{43}$$

holds for $1 \leq i, j, k, l \leq n$, $i < j$, $k < l$.

Secondly, suppose that $1 \leq i, j, k, l \leq n$, $i < j$, $k < l$. If i, j, k, l are different, then

$$\begin{aligned}\{x_{ij}, y_{kl}\}_\Lambda &= -\{y_{ij}|_{\mu_j - \pi/2}, y_{kl}|_{\mu_j - \pi/2}\}_\Lambda \\ &= -(-\delta_{li}\, y_{jk} + \delta_{jl}\, y_{ik} + \delta_{ik}\, y_{jl} - \delta_{jk}\, y_{il})|_{\mu_j - \pi/2} \\ &= 0 \\ &= \delta_{il}\, x_{jk} - \delta_{jk}\, x_{il} + \delta_{jl}\, x_{ik} - \delta_{ik}\, x_{jl}.\end{aligned}$$

If $k = j$, $i \neq l$, then

$$\begin{aligned}\{x_{ij}, y_{jl}\}_\Lambda &= \{y_{ij}|_{\mu_j - \pi/2}, y_{jl}|_{\mu_j - \pi/2,\ \mu_l - \pi/2}\}_\Lambda \\ &= (-\delta_{li}\, y_{jj} + \delta_{jl}\, y_{ij} + \delta_{ij}\, y_{jl} - \delta_{jj}\, y_{il})|_{\mu_j - \pi/2,\ \mu_l - \pi/2} \\ &= (-\delta_{jj}\, y_{il})|_{\mu_j - \pi/2,\ \mu_l - \pi/2} \\ &= -\delta_{jj}\, y_{il}|_{\mu_j,\ \mu_l - \pi/2} \\ &= -\delta_{jj}\, x_{il}|_{\mu_j,\ \mu_l} \\ &= \delta_{il}\, x_{jk} - \delta_{jk}\, x_{il} + \delta_{jl}\, x_{ik} - \delta_{ik}\, x_{jl}.\end{aligned}$$

If $l = j$, $i \neq k$, then

$$\begin{aligned}\{x_{ij}, y_{kj}\}_\Lambda &= \{y_{ij}|_{\mu_k - \pi/2,\ \mu_j - \pi/2}, y_{kj}|_{\mu_k - \pi/2,\ \mu_j - \pi/2}\}_\Lambda \\ &= (-\delta_{ji}\, y_{jk} + \delta_{jj}\, y_{ik} + \delta_{ik}\, y_{jj} - \delta_{jk}\, y_{ij})|_{\mu_k - \pi/2,\ \mu_j - \pi/2} \\ &= (\delta_{jj}\, y_{ik})|_{\mu_k - \pi/2,\ \mu_j - \pi/2} \\ &= \delta_{jl}\, y_{ik}|_{\mu_k - \pi/2,\ \mu_j} \\ &= \delta_{jl}\, x_{ik}|_{\mu_k,\ \mu_j} \\ &= \delta_{il}\, x_{jk} - \delta_{jk}\, x_{il} + \delta_{jl}\, x_{ik} - \delta_{ik}\, x_{jl}.\end{aligned}$$

Hence it was shown that

$$\{x_{ij}, y_{kl}\}_\Lambda = \delta_{il}\, x_{jk} - \delta_{jk}\, x_{il} + \delta_{jl}\, x_{ik} - \delta_{ik}\, x_{jl} \tag{44}$$

holds for $1 \leq i, j, k, l \leq n$, $i < j$, $k < l$.

Lastly, for $1 \leq i < j \leq n$,

$$\begin{aligned}\{x_{ij}, y_{ij}\}_\Lambda &= \{f_{ij}(\lambda)\cos(\mu_i - \mu_j + \phi_{ij}), f_{ij}(\lambda)\sin(\mu_i - \mu_j + \phi_{ij})\}_\Lambda \\ &= \{f_{ij}(\lambda), f_{ij}(\lambda)\sin(\mu_i - \mu_j + \phi_{ij})\}_\Lambda \cos(\mu_i - \mu_j + \phi_{ij}) \\ &\quad + f_{ij}(\lambda)\{\cos(\mu_i - \mu_j + \phi_{ij}), f_{ij}(\lambda)\sin(\mu_i - \mu_j + \phi_{ij})\}_\Lambda \\ &= \{f_{ij}(\lambda), \sin(\mu_i - \mu_j + \phi_{ij})\}_\Lambda f_{ij}(\lambda)\cos(\mu_i - \mu_j + \phi_{ij}) \\ &\quad + f_{ij}(\lambda)\{\cos(\mu_i - \mu_j + \phi_{ij}), f_{ij}(\lambda)\}_\Lambda \sin(\mu_i - \mu_j + \phi_{ij}) \\ &= \left(-\frac{\partial f_{ij}(\lambda)}{\partial \lambda^i}\frac{\partial \sin(\mu_i - \mu_j + \phi_{ij})}{\partial \mu_i} - \frac{\partial f_{ij}(\lambda)}{\partial \lambda^j}\frac{\partial \sin(\mu_i - \mu_j + \phi_{ij})}{\partial \mu_j}\right) \\ &\quad \times f_{ij}(\lambda)\cos(\mu_i - \mu_j + \phi_{ij}) \\ &\quad + \left(\frac{\partial \cos(\mu_i - \mu_j + \phi_{ij})}{\partial \mu_i}\frac{\partial f_{ij}(\lambda)}{\partial \lambda^i} + \frac{\partial \cos(\mu_i - \mu_j + \phi_{ij})}{\partial \mu_j}\frac{\partial f_{ij}(\lambda)}{\partial \lambda^j}\right) \\ &\quad \times f_{ij}(\lambda)\sin(\mu_i - \mu_j + \phi_{ij}) \\ &= \left(-\frac{\partial f_{ij}(\lambda)}{\partial \lambda^i}\cos(\mu_i - \mu_j + \phi_{ij}) + \frac{\partial f_{ij}(\lambda)}{\partial \lambda^j}\cos(\mu_i - \mu_j + \phi_{ij})\right)\end{aligned}$$

.

$$
\begin{aligned}
& \times f_{ij}(\lambda)\cos(\mu_i - \mu_j + \phi_{ij}) \\
& + \left(-\sin(\mu_i - \mu_j + \phi_{ij}) \frac{\partial f_{ij}(\lambda)}{\partial \lambda^i} + \sin(\mu_i - \mu_j + \phi_{ij}) \frac{\partial f_{ij}(\lambda)}{\partial \lambda^j} \right) \\
& \times f_{ij}(\lambda)\sin(\mu_i - \mu_j + \phi_{ij}) \\
= & \left(-\frac{\partial f_{ij}(\lambda)}{\partial \lambda^i} + \frac{\partial f_{ij}(\lambda)}{\partial \lambda^j} \right) f_{ij}(\lambda)\cos^2(\mu_i - \mu_j + \phi_{ij}) \\
& + \left(-\frac{\partial f_{ij}(\lambda)}{\partial \lambda^i} + \frac{\partial f_{ij}(\lambda)}{\partial \lambda^j} \right) f_{ij}(\lambda)\sin^2(\mu_i - \mu_j + \phi_{ij}) \\
= & \left(-\frac{\partial f_{ij}(\lambda)}{\partial \lambda^i} + \frac{\partial f_{ij}(\lambda)}{\partial \lambda^j} \right) f_{ij}(\lambda).
\end{aligned}
$$

By the virtue of the condition (28),

$$
\begin{aligned}
\{x_{ij}, y_{ij}\}_\Lambda &= \left(-\sqrt{\frac{\epsilon + \lambda^j}{\epsilon + \lambda^i}} + \sqrt{\frac{\epsilon + \lambda^i}{\epsilon + \lambda^j}} \right) f_{ij}(\lambda) \\
&= \left(-2|\epsilon + \lambda^j| + 2|\epsilon + \lambda^i| \right) \\
&= 2\left(|\epsilon + \lambda^i| - |\epsilon + \lambda^j| \right) \\
&= 2\left(\epsilon + \lambda^i - \epsilon - \lambda^j \right) \\
&= 2\left(\lambda^i - \lambda^j \right) \\
&= 2\left(t_i - t_j \right).
\end{aligned}
$$

Thus we obtain for $1 \leq i < j \leq n$,

$$\{x_{ij}, y_{ij}\}_\Lambda = 2\,(t_i - t_j). \tag{45}$$

The proof is completed.

Ⅳ. Discussion

In the context of quantum-mechanical probability[(4)], what does the symplectic manifold Λ or Δ_ε mean? An element in $\mathbf{u}(n)$ multiplied by $\sqrt{-1}$ is regarded as an observable in an n-dimensional quantum-mechanical probability space. This correspondence suggests that some distribution on Δ_ε corresponds to a quantum-mechanical state. Under this correspondence, we can see in the following that a quantum-mechanical pure state corresponds to a point in Δ_0.

Let $|\varphi>$ be a quantum-mechanical pure state[(4)] in the n-dimensional quantum probability space. $|\varphi>$ can be expressed as

$$|\varphi\rangle = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}, \varphi_i \in \mathbf{C}, \tag{46}$$

where

$$\sum_{i=1}^{n} |\varphi_i|^2 = 1. \tag{47}$$

By the polar decomposition of a complex number,

$$\varphi_i = |\varphi_i| \left(\cos(\arg\varphi_i) + \sqrt{-1}\sin(\arg\varphi_i)\right). \tag{48}$$

Then

$$\overline{\varphi_j}\varphi_i = |\varphi_i|\ |\varphi_j|\ \left[\cos(\arg\varphi_i - \arg\varphi_j) + \sqrt{-1}\sin(\arg\varphi_i - \arg\varphi_j)\right]. \tag{49}$$

Using this, we calculate expectation values of JT_i, JX_{ij}, JY_{ij}.

$$\mathrm{Tr}\,(|\varphi\rangle\langle\varphi|JT_i) = \langle\varphi|JT_i|\varphi\rangle = \sqrt{-1}\langle\varphi|E_{ii}|\varphi\rangle = \sqrt{-1}|\varphi_i|^2. \tag{50}$$

$$\begin{aligned}\mathrm{Tr}\,(|\varphi\rangle\langle\varphi|JX_{ij}) &= \langle\varphi|JX_{ij}|\varphi\rangle \\ &= \sqrt{-1}\langle\varphi|E_{ij} + E_{ji}|\varphi\rangle \\ &= \sqrt{-1}(\overline{\varphi_i}\varphi_j + \overline{\varphi_j}\varphi_i) \\ &= 2\sqrt{-1}\Re(\overline{\varphi_j}\varphi_i) \\ &= 2\sqrt{-1}|\varphi_i|\ |\varphi_j|\ \cos(\arg\varphi_i - \arg\varphi_j).\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}\,(|\varphi\rangle\langle\varphi|JY_{ij}) &= \langle\varphi|JY_{ij}|\varphi\rangle \\ &= \langle\varphi| - E_{ij} + E_{ji}|\varphi\rangle \\ &= (-\overline{\varphi_i}\varphi_j + \overline{\varphi_j}\varphi_i) \\ &= 2\sqrt{-1}\Im(\overline{\varphi_j}\varphi_i) \\ &= 2\sqrt{-1}|\varphi_i|\ |\varphi_j|\ \sin(\arg\varphi_i - \arg\varphi_j).\end{aligned}$$

Put $\varepsilon = 0$. For $|\varphi>$, we define a point $(\lambda_\varphi, \mu_\varphi) \in \Delta_0$ by

$$(\lambda_\varphi)^i := |\varphi_i|^2, \quad i = 1, \dots, n, \tag{51}$$

$$(\mu_\varphi)_i := \arg\varphi_i, \quad i = 1, \dots, n. \tag{52}$$

And we put

$$\theta_i = 0,\ i = 1, \dots, n. \tag{53}$$

Then the corresponding distribution s_φ on Δ_0 is gven by

$$s_\varphi(\lambda, \mu) := \sqrt{-1}\delta(\lambda - \lambda_\varphi)\delta(\mu - \mu_\varphi), \tag{54}$$

because

$$\int_\Lambda d\lambda d\mu\ s_\varphi t_i = \mathrm{Tr}\,(|\varphi\rangle\langle\varphi|JT_i), \tag{55}$$

$$\int_\Lambda d\lambda d\mu\ s_\varphi x_{ij} = \mathrm{Tr}\,(|\varphi\rangle\langle\varphi|JX_{ij}), \tag{56}$$

$$\int_\Lambda d\lambda d\mu\ s_\varphi y_{ij} = \mathrm{Tr}\,(|\varphi\rangle\langle\varphi|JY_{ij}) \tag{57}$$

hold, in other words, the corresponding expectation values coincide. Thus the support of s_φ consists of one point $(\lambda_\varphi, \mu_\varphi)$; we can say that a point in Δ_0 corresponds to a quantum-mechanical pure state. This means that the symplectic manifold Λ cannot be interpreted as a phase space of classical physical states.

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[Abstract]

Representations of the Lie Algebra of $U(n, \mathbf{C})$ in Terms of Poisson Brackets

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A faithful representation of the Lie algebra of *n*-dimensional unitary group is constructed by a Lie algebra of functions on a 2n-dimensional symplectic manifold. A Poisson bracket of two functions on the symplectic manifold is defined through the symplectic structure in a standard manner. In this representation, the Lie products are given by the Poisson brackets. It is also discussed what the symplectic manifold means when this representation is applied to the quantum-mechanical probability theory. It is shown that a point of the symplectic manifold can be interpreted as a pure quantum-mechanical state by corresponding each of the coordinates of the point to the square of the absolute value or the argument of each component of the pure state vector.

Key words : unitary group, Lie algebra, representation, Poisson bracket